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Regularity of solutions to a model for solid–solid phase transitions driven by configurational forces

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ABSTRACT

In a previous work, we prove the existence of weak solutions to an initial-boundary value problem, with $H^1(\Omega)$ initial data, for a system of partial differential equations, which consists of the equations of linear elasticity and a nonlinear, degenerate parabolic equation of second order. This problem models the behavior in time of materials with martensitic phase transitions. This model with diffusive phase interfaces was derived from a model with sharp interfaces, whose evolution is driven by configurational forces, and can be regarded as a regularization of that model. Assuming in this article the initial data is in $H^2(\Omega)$, we investigate the regularity of weak solutions that is difficult due to the gradient term which plays a role of a weight. Our proof, in which the difficulties are caused by the weight in the principle term, is only valid in one space dimension.

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1. Introduction

Many inhomogeneous systems can be characterized by domains of different phases separated by a distinct interface [13,14]. Driven out of equilibrium, their dynamics result in the evolution of those interfaces which might develop into structures (compositional and structural inhomogeneities) with characteristic length scales at the nano-, micro- or meso-scale. To a large extent, the material properties of such systems are determined by those structures of small-scale. It is thus important to understand precisely the mechanisms that drive the evolution of those structures. In this article we are interested in a model for the evolution, driven by configurational forces, of microstructures in elastically deformable solids. Materials microstructures may consist of spatially distributed phases of different compositions and/or crystal structures, grains of different orientations, domains of different structural variants, domains of different electrical or magnetic polarizations, and structural defects, see e.g. [12]. These structural features usually have an intermediate mesoscopic length scale in the range of nanometers to microns. The size, shape, and spatial arrangement of the local structural features in a microstructure play a critical role in determining the physical properties of a material. Because of the complex and nonlinear nature of microstructure evolution, numerical approaches are often employed.

There are two main types of modeling for the evolution of microstructures. In the conventional approach, the regions separating the domains are treated as mathematically sharp interfaces. The local interfacial velocity is then determined as part of the boundary conditions, or is calculated from the driving force for interface motion and the interfacial mobility. This approach requires the explicit tracking of the interface positions. Such an interface-tracking approach can be successful in one-dimensional systems, however it will be impractical for complicated three-dimensional microstructures. Therefore, during the past decades, another approach has been invented, namely, the phase-field approach in which the interface

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is not of zero thickness, instead an interfacial region with thickness of certain order of a small regularization parameter. Though it is still a young discipline in condensed matter physics, this approach has emerged to be one of the most powerful methods for modeling the evolution of microstructures. It can be traced back the theory of diffuse-interface description, which is developed, independently, more than a century ago by van der Waals [21] and some half century ago by Cahn and Hilliard [10].

The two well-known models for temporal evolution of microstructures are the Cahn–Hilliard/Allen–Cahn equations corresponding, respectively, to the cases that the order parameter is conserved and not conserved. These phase field models describe microstructure phenomena at the mesoscale, and one suitable limit of it may be the corresponding sharp- or thin-interface descriptions. In this article we study a model for the behavior in time of materials with diffusionless phase transitions. The model has diffusive interfaces and consists of the partial differential equations of linear elasticity coupled to a quasilinear, non-uniformly parabolic equation of second order that differs from the Allen–Cahn equation (the Cahn–Hilliard equation in the case that the order parameter is conserved) by a gradient term. It is derived in [1,3] from a sharp interface model for diffusionless phase transitions and can be considered to be a regularization of that model. To verify the validity of the new model, mathematical analysis has been carried out for the existence of weak solutions to initial boundary value problems in one space dimension [2,4,8,6,22], the existence of spherically symmetric solutions [7,18], the motion of interfaces [5], and the existence of traveling waves [16]. In the present article, the existence and regularity of weak solutions to an initial-boundary value problem will be studied. We first formulate this initial-boundary value problem in the three-dimensional case, reduce it to the one-dimensional case and prove the existence of weak solutions to this one-dimensional problem, then study the regularity of weak solutions by assuming that the initial data is in $H^2(\Omega)$.

Let $\Omega \subset \mathbb{R}^3$ be an open set. It represents the material points of a solid body. The different phases are characterized by the order parameter $S(t,x) \in \mathbb{R}$. A value of $S(t,x)$ near to zero indicates that the material is in the matrix phase at the point $x \in \Omega$ at time t, a value near to one indicates that the material is in the second phase. The other unknowns are the displacement $u(t,x) \in \mathbb{R}^3$ of the material point x at time t and the Cauchy stress tensor $T(t,x) \in \mathcal{S}^3$, where $\mathcal{S}^3$ denotes the set of symmetric 3×3-matrices. The unknowns must satisfy the quasi-static equations

$$-\operatorname{div}_x T(t,x) = b(t,x), \tag{1.1}$$

$$T(t,x) = D\big(\varepsilon\big(\nabla_x u(t,x)\big) - \bar{\varepsilon} S(t,x)\big), \tag{1.2}$$

$$S_t(t,x) = -c\big(\psi_S\big(\varepsilon\big(\nabla_x u(t,x)\big), S(t,x)\big) - \nu \Delta_x S(t,x)\big)\big|\nabla_x S(t,x)\big| \tag{1.3}$$

for $(t,x) \in (0,\infty) \times \Omega$. The boundary and initial conditions are

$$u(t,x) = \gamma(t,x), \qquad S(t,x) = 0, \quad (t,x) \in [0,\infty) \times \partial\Omega, \tag{1.4}$$

$$S(0,x) = S_0(x), \quad x \in \Omega. \tag{1.5}$$

Here $\nabla_x u$ denotes the 3×3-matrix of first order derivatives of u, the deformation gradient, $(\nabla_x u)^T$ denotes the transposed matrix and

$$\varepsilon(\nabla_x u) = \frac{1}{2}\big(\nabla_x u + (\nabla_x u)^T\big)$$

is the strain tensor. $\bar{\varepsilon} \in \mathcal{S}^3$ is a given matrix, the misfit strain, and $D : \mathcal{S}^3 \to \mathcal{S}^3$ is a linear, symmetric, positive definite mapping, the elasticity tensor. In the free energy

$$\psi(\varepsilon, S) = \frac{1}{2}\big(D(\varepsilon - \bar{\varepsilon}S)\big) \cdot (\varepsilon - \bar{\varepsilon}S) + \hat{\psi}(S) \tag{1.6}$$

we choose for $\hat{\psi} \in C^2(\mathbb{R}, [0,\infty))$ a double well potential with minima at $S = 0$ and $S = 1$. The scalar product of two matrices is $A \cdot B = \sum a_{ij} b_{ij}$. Also, ψ_S is the partial derivative, $c > 0$ is a constant and ν is a small positive constant. Given are the volume force $b : [0,\infty) \times \Omega \to \mathbb{R}^3$ and the data $\gamma : [0,\infty) \times \partial\Omega \to \mathbb{R}^3$, $S_0 : \Omega \to \mathbb{R}$.

This completes the formulation of the initial-boundary value problem. Eqs. (1.1) and (1.2) differ from the system of linear elasticity only by the term $\bar{\varepsilon}S$. The evolution equation (1.3) for the order parameter S is non-uniformly parabolic because of the term $\nu \Delta S |\nabla_x S|$. Since this initial-boundary value problem is derived from a sharp interface model, to verify that it is indeed a diffusive interface model regularizing the sharp interface model, it must be shown that Eqs. (1.1)–(1.5) with positive ν have solutions which exist global in time and is more regular if the initial data is more regular, and that these solutions tend to solutions of the sharp interface model for $\nu \to 0$. This would also be a method to prove existence of solutions to the original sharp interface model.

In this article we show that in one space dimension the initial-boundary value problem has solutions, and study the regularity of these weak solutions with $H^2(\Omega)$ initial data. Whether solutions in three space dimensions exist and whether these solutions converge to a solution of the sharp interface model for $\nu \to 0$ is still an open problem to be investigated later. The model and therefore the existence result is of interest not only in three dimensions but also in one space dimension. Yet, we believe that this one-dimensional existence result can also be a basic building block in an existence proof for higher space dimensions, which is sketched at the end of this introduction.

Related to our investigations is the model for diffusion dominated phase transformations obtained by coupling the elasticity equations (1.1), (1.2) with the Cahn–Hilliard equation. This model has recently been studied in [9,11,15].

Statement of the main result. We now assume that all functions only depend on the variables x_1 and t, and, to simplify the notation, denote x_1 by x. The set $\Omega = (a, d)$ is a bounded open interval with constants $a < d$. We write $Q_{T_e} := (0, T_e) \times \Omega$, where T_e is a positive constant, and define

$$(v, \varphi)_Z = \int\limits_Z v(y)\varphi(y)\,dy,$$

for $Z = \Omega$ or $Z = Q_{T_e}$. If v is a function defined on Q_{T_e} we denote the mapping $x \mapsto v(t, x)$ by $v(t)$. If no confusion is possible we sometimes drop the argument t and write $v = v(t)$. We still allow that the material points can be displaced in three directions, hence $u(t, x) \in \mathbb{R}^3$, $T(t, x) \in \mathcal{S}^3$ and $S \in \mathbb{R}$. If we denote the first column of the matrix $T(t, x)$ by $T_1(t, x)$ and set

$$\varepsilon(u_x) = \frac{1}{2}\big((u_x, 0, 0) + (u_x, 0, 0)^T\big) \in \mathcal{S}^3,$$

then with these definitions Eqs. (1.1)–(1.3) in the case of one space dimension can be written in the form

$$-T_{1x} = b, \tag{1.7}$$

$$T = D\big(\varepsilon(u_x) - \bar{\varepsilon}S\big), \tag{1.8}$$

$$S_t = c\big(T \cdot \bar{\varepsilon} - \hat{\psi}'(S) + \nu S_{xx}\big)|S_x|, \tag{1.9}$$

which must be satisfied in Q_{T_e}. Here we have inserted $\psi_S(\varepsilon, S) = -T \cdot \bar{\varepsilon} + \hat{\psi}'(S)$. Since Eqs. (1.7), (1.8) are linear, the inhomogeneous Dirichlet boundary condition for u can be reduced in the standard way to the homogeneous condition. For simplicity we thus assume that $\gamma = 0$. The initial and boundary conditions therefore are

$$u(t, x) = 0, \quad (t, x) \in (0, T_e) \times \partial\Omega, \tag{1.10}$$

$$S(t, x) = 0, \quad (t, x) \in (0, T_e) \times \partial\Omega, \tag{1.11}$$

$$S(0, x) = S_0(x), \quad x \in \Omega. \tag{1.12}$$

To define weak solutions of this initial-boundary value problem we use the fact that $\frac{1}{2}(|y|y)' = |y|$ to show that Eq. (1.9) is equivalent to

$$S_t - c\nu\frac{1}{2}\big(|S_x|S_x\big)_x - c\big(T \cdot \bar{\varepsilon} - \hat{\psi}'(S)\big)|S_x| = 0. \tag{1.13}$$

Definition 1.1. Let $b \in L^\infty(0, T_e, L^2(\Omega))$, $S_0 \in L^\infty(\Omega)$. A function (u, T, S) with

$$u \in L^\infty\big(0, T_e; W_0^{1,\infty}(\Omega)\big), \tag{1.14}$$

$$T \in L^\infty(Q_{T_e}), \tag{1.15}$$

$$S \in L^\infty(Q_{T_e}) \cap L^\infty\big(0, T_e; H_0^1(\Omega)\big), \tag{1.16}$$

is a weak solution to problem (1.7)–(1.12), if Eqs. (1.7)–(1.8) with (1.10) are satisfied in the weak sense, and if for all $\varphi \in C_0^\infty((-\infty, T_e) \times \Omega)$

$$(S, \varphi_t)_{Q_{T_e}} - c\nu\frac{1}{2}\big(|S_x|S_x, \varphi_x\big)_{Q_{T_e}} + c\big(\big(T \cdot \bar{\varepsilon} - \hat{\psi}'(S)\big)|S_x|, \varphi\big)_{Q_{T_e}} + \big(S_0, \varphi(0)\big)_\Omega = 0. \tag{1.17}$$

We assume that the nonlinearity $\hat{\psi}(S)$ is a *smooth* double-well potential, namely, $\hat{\psi}(S)$ satisfies the following:

Assumption for $\hat{\psi}(S)$:

$$\begin{aligned} &\hat{\psi}(S) \in C^\infty(\mathbb{R}) \text{ is a double-well potential which has two local minima at} \\ &S_- \text{ and } S_+ \text{ with } S_- < S_+ \text{ and one local maximum at } S_* \text{ with } S_- < S_* < S_+, \\ &\text{and satisfies } \hat{\psi}'(S) > 0 \text{ for } S_- < S < S_* \text{ and } \hat{\psi}'(S) < 0 \text{ for } S_* < S < S_+. \end{aligned} \tag{1.18}$$

One typical example is: $\hat{\psi}(S) = (S(1 - S))^2$ with $S_- = 0$, $S_+ = 1$.

Now we are in a position to state the main result of this article.

Theorem 1.2. *Assume that $\hat{\psi}(S)$ satisfies* (1.18)*. Then for all $S_0 \in H_0^1(\Omega) \cap H^2(\Omega)$ and $b \in C(\bar{Q}_{T_e})$ with $b_t \in C(\bar{Q}_{T_e})$, there exists a weak solution (u, T, S) to problem* (1.7)–(1.12)*, which in addition to* (1.14)–(1.17) *satisfies*

$$u_t \in C\big([0, T_e]; H^1(\Omega)\big), \qquad T_t \in C\big([0, T_e]; L^2(\Omega)\big) \tag{1.19}$$

and

$$S_t \in L^\infty\big(0, T_e; L^2(\Omega)\big), \qquad |S_x| S_x \in L^\infty\big(0, T_e; H^1(\Omega)\big), \qquad S_x \in L^\infty(Q_{T_e}). \tag{1.20}$$

The remaining sections are devoted to the proof of this theorem. The main difficulty in the proof stems from the fact that the coefficient $\nu|S_x|$ of the highest order derivative S_{xx} in Eq. (1.9) is not bounded away from zero and that it is not differentiable with respect to S_x.

To prove Theorem 1.2 we therefore consider in Section 2 a modified initial-boundary value problem which consists of (1.7), (1.8), (1.10)–(1.12) and the equation

$$S_t - c\nu |S_x|_\kappa S_{xx} - c\big(T \cdot \bar{\varepsilon} - \hat{\psi}'(S)\big)\big(|S_x|_\kappa - \kappa\big) = 0, \quad x \in \Omega,\ t > 0 \tag{1.21}$$

with a constant $\kappa > 0$. Here we use the notation

$$|p|_\kappa := \sqrt{\kappa^2 + |p|^2}. \tag{1.22}$$

Since (1.21) is a uniformly (for $|S_x| \leqslant M$, M is a positive constant) parabolic equation we can use a standard theorem to conclude that the modified initial-boundary value problem has a sufficiently smooth solution $(u^\kappa, T^\kappa, S^\kappa)$. For this solution we derive in Section 3 a priori estimates independent of $\kappa \in (0, 1]$.

The function $|p|$ is smoothed by $|p|_\kappa$ in (1.22) which is different from that in [2], thus we also prove the existence of weak solutions, though our main concern of this article is the regularity of solutions. To select a subsequence converging to a solution for $\kappa \to 0$ we need a compactness result. However, our a priori estimates of S_{xx} depend on a weight $|S_x|_\kappa$, and are not strong enough to show that the sequence S_x^κ is compact; instead, we can only show that the sequence $\int_0^{S_x^\kappa} |y|^{\frac{1}{2}}\, dy$ is compact, from which we conclude that these is a subsequence of S_x^κ that converges almost everywhere, thereby proving the existence. For the compactness proof in Section 4 we use the compact Sobolev embedding theorem, and don't need the Aubin–Lions Lemma or its generalized form of this lemma given by Roubíček [19], Simon [20], which plays a crucial role in the article [2].

In the proof of the regularity estimates, we differentiate Eq. (1.21) with respect to t. Thus a term like $(|S_x|_\kappa)_t \psi_S$ appears, which cannot be absorbed by the a priori estimates with a weight $|S_x|_\kappa$. To overcome this difficulty, we derive a type of estimate (see (3.6)) with a reciprocal weight $|S_x|_\kappa^{-1}$. This is possible due to the special structure of the model studied here. The Allen–Cahn model does not possesses such a structure, and therefore our technique does not work for that model.

The method of proof is limited to one space dimension, since for the a priori estimates it is crucial that the term $|S_x| S_{xx}$ in (1.9) can be written in the divergence form $\frac{1}{2}(|S_x| S_x)_x$. In the higher dimensional case the corresponding term $|\nabla_x S| \Delta_x S$ cannot be rewritten in this way, whence the multi-dimensional problem is still open.

2. Existence of solutions to the modified problem

In this section, we study the modified initial-boundary value problem and show that it has a Hölder continuous classical solution. To formulate this problem, let $\chi \in C_0^\infty(\mathbb{R}, [0, \infty))$ satisfy $\int_{-\infty}^{\infty} \chi(t)\, dt = 1$. For $\kappa > 0$, we set

$$\chi_\kappa(t) := \frac{1}{\kappa} \chi\left(\frac{t}{\kappa}\right),$$

and for $S \in L^\infty(Q_{T_e}, \mathbb{R})$ we define

$$(\chi_\kappa * S)(t, x) = \int_0^{T_e} \chi_\kappa(t - s) S(s, x)\, ds. \tag{2.1}$$

The modified initial-boundary value problem consists of the equations

$$-T_{1x} = b, \tag{2.2}$$

$$T = D\big(\varepsilon(u_x) - \bar{\varepsilon} \chi_\kappa * S\big), \tag{2.3}$$

$$S_t = c\nu |S_x|_\kappa S_{xx} + c\big(T \cdot \bar{\varepsilon} - \hat{\psi}'(S)\big)\big(|S_x|_\kappa - \kappa\big), \tag{2.4}$$

which must hold in Q_{T_e}, and of the boundary and initial conditions

$$u(t,x)=0, \quad (t,x)\in(0,T_e)\times\partial\Omega, \tag{2.5}$$

$$S(t,x)=0, \quad (t,x)\in(0,T_e)\times\partial\Omega, \tag{2.6}$$

$$S(0,x)=S_0(x), \quad x\in\Omega. \tag{2.7}$$

We need to introduce some function spaces before formulating an existence theorem for this problem. For nonnegative integers m,n and a real number $\alpha\in(0,1)$ we denote by $C^{m+\alpha}(\bar\Omega)$ the space of m-times differentiable functions on $\bar\Omega$, whose m-th derivative is Hölder continuous with exponent α. The space $C^{\alpha,\alpha/2}(\bar Q_{T_e})$ consists of all functions on $\bar Q_{T_e}$, which are Hölder continuous in the parabolic distance

$$\mathrm{d}\big((t,x),(s,y)\big):=\sqrt{|t-s|+|x-y|^2}.$$

$C^{m,n}(\bar Q_{T_e})$ and $C^{m+\alpha,n+\alpha/2}(\bar Q_{T_e})$, respectively, are the spaces of functions, whose x-derivatives up to order m and t–derivatives up to order n belong to $C(\bar Q_{T_e})$ or to $C^{\alpha,\alpha/2}(\bar Q_{T_e})$, respectively.

Theorem 2.1. *Let $\nu,\kappa>0$, $T_e>0$. Suppose that the function $b\in C(\bar Q_{T_e})$ has the derivative $b_t\in C(\bar Q_{T_e})$ and that the initial data $S_0\in C^{2+\alpha}(\bar\Omega)$ satisfy $S_0|_{\partial\Omega}=S_{0,x}|_{\partial\Omega}=S_{0,xx}|_{\partial\Omega}=0$. Then there is a solution*

$$(u,T,S)\in C^{2,1}(\bar Q_{T_e})\times C^{1,1}(\bar Q_{T_e})\times C^{2+\alpha,1+\alpha/2}(\bar Q_{T_e})$$

of the modified initial-boundary value problem (2.2)–(2.7). *This solution satisfies $S_{tx}\in L^2(Q_{T_e})$ and*

$$\max_{\bar Q_{T_e}}|S|\leqslant\max_{\bar\Omega}|S_0|. \tag{2.8}$$

Proof. In [3] it is shown that the unique solution to the linear elliptic problem (2.2)–(2.3), with (2.5) and given S, is given by

$$u(t,x)=u^*\left(\int_a^x(\chi_\kappa*S)(t,y)\,dy-\frac{x-a}{d-a}\int_a^d(\chi_\kappa*S)(t,y)\,dy\right)+w(t,x), \tag{2.9}$$

$$T(t,x)=D\big(\varepsilon^*-\bar\varepsilon\big)(\chi_\kappa*S)(t,x)-\frac{D\varepsilon^*}{d-a}\int_a^d(\chi_\kappa*S)(t,y)\,dy+\sigma(t,x), \tag{2.10}$$

where $u^*\in\mathbb{R}^3$, $\varepsilon^*\in\mathcal{S}^3$ are suitable constants only depending on $\bar\varepsilon$ and D, and where for every $t\in[0,T_e]$ the function $(w(t),\sigma(t)):\Omega\to\mathbb{R}^3\times\mathcal{S}^3$ is the solution to the boundary value problem

$$-\sigma_{1x}(t)=b(t), \qquad \sigma(t)=D\varepsilon\big(w_x(t)\big), \qquad w(t)_{|\partial\Omega}=0.$$

Since by assumption b and b_t belong to $C(\bar Q_{T_e})$, it follows that $(w,\sigma)\in C^{2,1}(\bar Q_{T_e})\times C^{1,1}(\bar Q_{T_e})$. We insert (2.10) into (2.4) and obtain the equation

$$S_t=a_1(S_x)S_{xx}+a_2\left(t,x,S,S_x,\chi_\kappa*S,\frac{1}{d-a}\int_a^d(\chi_\kappa*S)(t,y)\,dy\right) \tag{2.11}$$

in Q_{T_e}, where

$$a_1(p)=c\nu|p|_\kappa$$

and

$$a_2(t,x,S,p,r,s)=c\big(\bar\varepsilon\cdot D\big(\varepsilon^*-\bar\varepsilon\big)r-\bar\varepsilon\cdot D\varepsilon^*s+\bar\varepsilon\cdot\sigma(t,x)-\hat\psi'(S)\big)\big(|p|_\kappa-\kappa\big).$$

Eqs. (2.11), (2.6) and (2.7) form an initial-boundary value problem with nonlocal terms, which is equivalent to the problem (2.2)–(2.7). We can apply [17, Theorem 2.9, p. 23], with a slight modification, to (2.11) and conclude the existence of classical solution to the modified problem and estimate (2.8) holds. For the details, we refer to [2]. □

3. A priori estimates

In this section we establish a priori estimates for solutions of the modified problem, which are uniform with respect to $\kappa \in (0, 1]$. We remark that the estimates in Lemma 3.1 and some in Corollary 3.2, though stated in the one-dimensional case, can be generalized to higher space dimensions.

In what follows we assume that

$$0 < \kappa \leqslant 1, \tag{3.1}$$

since we consider the limit $\kappa \to 0$. The $L^2(\Omega)$-norm is denoted by $\|\cdot\|$, and the letter C stands for varies positive constants independent of κ, while may depend on ν.

We start by constructing a family of approximate solutions to the modified problem. To this end let T_e be a fixed positive number and choose for every κ a function $S_0^\kappa \in C_0^\infty(\Omega)$ such that

$$\left\| S_0^\kappa - S_0 \right\|_{H_0^1(\Omega)\cap H^2(\Omega)} \to 0, \quad \kappa \to 0, \tag{3.2}$$

where $S_0 \in H_0^1(\Omega) \cap H^2(\Omega)$ are the initial data given in Theorem 1.2. We insert for S_0 in (2.7) the function S_0^κ and choose for b in (2.2) the function given in Theorem 1.1. These functions satisfy the assumptions of Theorem 2.1, hence there is a solution $(u^\kappa, T^\kappa, S^\kappa)$ of the modified problem (2.2)–(2.7), which exists in Q_{T_e}. The inequality (2.8) and Sobolev's embedding theorem yield for this solution

$$\sup_{0<\kappa\leqslant 1} \left\| S^\kappa \right\|_{L^\infty(Q_{T_e})} \leqslant \sup_{0<\kappa\leqslant 1} \left\| S_0^\kappa \right\|_{L^\infty(\Omega)} \leqslant C. \tag{3.3}$$

Remembering that σ in (2.10) belongs to $C^{1,1}(\bar{Q}_{T_e})$, we conclude from (3.3) that also

$$\max_{\bar{Q}_{T_e}} \left| c\left(T^\kappa \cdot \bar{\varepsilon} - \hat{\psi}'\left(S^\kappa\right)\right) \right| \leqslant C. \tag{3.4}$$

Lemma 3.1. *There hold for any $t \in [0, T_e]$*

$$\left\| S_x^\kappa(t) \right\|^2 + c\nu \int_0^t \int_\Omega \left| S_x^\kappa \right|_\kappa \left| S_{xx}^\kappa \right|^2 dx\, d\tau \leqslant C, \tag{3.5}$$

$$\int_0^t \int_\Omega \frac{|S_t^\kappa|^2}{|S_x^\kappa|_\kappa} dx\, d\tau \leqslant C. \tag{3.6}$$

Proof. Invoking $S_{tx}^\kappa \in L^2(Q_{T_e})$, by Theorem 2.1, which yields that for almost all t

$$\frac{1}{2}\frac{d}{dt}\left\| S_x^\kappa(t) \right\|^2 = \int_\Omega S_x^\kappa(t) S_{xt}^\kappa(t)\, dx.$$

Making use of this relation and estimate (3.4), multiplying (2.4) by $-S_{xx}^\kappa$ and integrating it with respect to x, and taking the boundary condition (2.6) into account, we obtain that for almost all t

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left\| S_x^\kappa \right\|^2 + c\nu \int_\Omega \left| S_x^\kappa \right|_\kappa \left| S_{xx}^\kappa \right|^2 dx \\
&\quad = \int_\Omega c\left(\hat{\psi}'\left(S^\kappa\right) - T^\kappa \cdot \bar{\varepsilon}\right)\left(\left|S_x^\kappa\right|_\kappa - \kappa\right) S_{xx}^\kappa\, dx \leqslant C \int_\Omega \left(\left|S_x^\kappa\right|_\kappa + \kappa\right)\left|S_{xx}^\kappa\right| dx \\
&\quad = C \int_\Omega \left|S_x^\kappa\right|_\kappa^{\frac{1}{2}} \left|S_x^\kappa\right|_\kappa^{\frac{1}{2}} \left|S_{xx}^\kappa\right| dx + C \int_\Omega \kappa \left|S_{xx}^\kappa\right| dx \leqslant \frac{c\nu}{4} \int_\Omega \left|S_x^\kappa\right|_\kappa \left|S_{xx}^\kappa\right|^2 dx + \frac{c\nu}{4}\kappa \left\|S_{xx}^\kappa\right\|^2 + \frac{C}{\nu} \int_\Omega \left(\left|S_x^\kappa\right|_\kappa\right)^2 dx + C_\nu.
\end{aligned} \tag{3.7}$$

Splitting the second term on the left-hand side of (3.7) into two equal terms and subtracting the term $\frac{c\nu}{4}\int_\Omega |S_x^\kappa|_\kappa |S_{xx}^\kappa|^2\, dx$ and $\frac{c\nu}{4}\kappa \|S_{xx}^\kappa\|^2$ on both sides of this inequality, and using Gronwall's Lemma we derive (3.5) from the resulting estimate, noting also (3.2) and $\kappa \leqslant |S_x^\kappa|_\kappa$.

To derive (3.6), we multiply (2.4) by $S_t^\kappa |S_x^\kappa|_\kappa^{-1}$ and integrate the resulting equation with respect to x to get

$$0=\int_\Omega \frac{(S_t^\kappa)^2}{|S_x^\kappa|_\kappa}\,dx-c\int_\Omega \big(\nu S_{xx}^\kappa-\psi_S\big)S_t^\kappa\,dx+c\int_\Omega \frac{\kappa\psi_S}{|S_x^\kappa|_\kappa}S_t^\kappa\,dx=\int_\Omega \frac{(S_t^\kappa)^2}{|S_x^\kappa|_\kappa}\,dx+I_1+I_2. \tag{3.8}$$

Invoking the formula $\psi_S=-T\cdot\bar\varepsilon+\hat\psi'(S)$ and the boundary conditions, and using integration by parts we have

$$I_1=c\frac{d}{dt}\int_\Omega\left(\frac{\nu}{2}\big|S_x^\kappa\big|^2+\hat\psi\big(S^\kappa\big)\right)dx-c\int_\Omega T^\kappa\cdot\bar\varepsilon S_t^\kappa\,dx=c\frac{d}{dt}\int_\Omega\left(\frac{\nu}{2}\big|S_x^\kappa\big|^2+\hat\psi\big(S^\kappa\big)\right)dx+J. \tag{3.9}$$

To deal with the term J, we use (3.4) to get

$$\begin{aligned}|J|&=c\left|\int_\Omega T^\kappa\cdot\bar\varepsilon\big|S_x^\kappa\big|_\kappa^{\frac12}\frac{S_t^\kappa}{|S_x^\kappa|_\kappa^{\frac12}}\,dx\right|\leqslant C\left(\int_\Omega\big|S_x^\kappa\big|_\kappa\,dx\right)^{\frac12}\left(\int_\Omega\frac{(S_t^\kappa)^2}{|S_x^\kappa|_\kappa}\,dx\right)^{\frac12}\\&\leqslant C\big(\big\|S_x^\kappa\big\|+1\big)^{\frac12}\left(\int_\Omega\frac{(S_t^\kappa)^2}{|S_x^\kappa|_\kappa}\,dx\right)^{\frac12}\leqslant\frac12\int_\Omega\frac{(S_t^\kappa)^2}{|S_x^\kappa|_\kappa}\,dx+C.\end{aligned} \tag{3.10}$$

Here we used the estimate (3.5) and the Cauchy–Schwarz and Young inequalities.

For I_2, we make use of Eq. (2.4) and write

$$I_2=c\int_\Omega\frac{\kappa\psi_S}{|S_x^\kappa|_\kappa}\big(c\nu\big|S_x^\kappa\big|_\kappa S_{xx}^\kappa-c\psi_S\big(\big|S_x^\kappa\big|_\kappa-\kappa\big)\big)\,dx=c^2\int_\Omega\left(\nu\kappa\psi_S S_{xx}^\kappa-\kappa(\psi_S)^2\frac{|S_x^\kappa|_\kappa-\kappa}{|S_x^\kappa|_\kappa}\right)dx. \tag{3.11}$$

By definition, one has $|S_x^\kappa|_\kappa\geqslant\kappa$ which implies $\frac{\kappa}{|S_x^\kappa|_\kappa}\leqslant 1$. So

$$|I_2|\leqslant C\int_\Omega\big(\kappa\big|S_{xx}^\kappa\big|+\big|S_x^\kappa\big|_\kappa+\kappa\big)\,dx\leqslant c\nu\kappa\big\|S_{xx}^\kappa\big\|^2+C_\nu\big(\big\|S_x^\kappa\big\|+1\big). \tag{3.12}$$

With the help of (3.5), (3.9)–(3.12), we integrate (3.8) with respect to t, then obtain

$$\frac12\int_0^t\int_\Omega\frac{(S_t^\kappa)^2}{|S_x^\kappa|_\kappa}\,dx+c\int_\Omega\left(\frac{\nu}{2}\big|S_x^\kappa\big|^2+\hat\psi\big(S^\kappa\big)\right)dx\leqslant C, \tag{3.13}$$

which implies (3.6). Thus we complete the proof of this lemma. □

Furthermore, we obtain

Corollary 3.2. *There holds for any $t\in[0,T_e]$*

$$\int_0^t\int_\Omega\big(\big|S_x^\kappa\big|_\kappa\big|S_{xx}^\kappa\big|\big)^{\frac43}\,dx\,d\tau\leqslant C, \tag{3.14}$$

$$\int_0^t\int_\Omega\big(\big|S_x^\kappa S_{xx}^\kappa\big|\big)^{\frac43}\,dx\,d\tau\leqslant C, \tag{3.15}$$

$$\int_0^t\left\|\int_0^{S_x^\kappa}|y|_\kappa\,dy\right\|_{W^{1,\frac43}(\Omega)}^{\frac43}d\tau\leqslant C, \tag{3.16}$$

$$\int_0^t\left\|\int_0^{S_x^\kappa}|y|_\kappa\,dy\right\|_{L^\infty(\Omega)}^{\frac43}d\tau\leqslant C, \tag{3.17}$$

$$\big\|\big|S_x^\kappa\big|S_x^\kappa\big\|_{L^{\frac43}(0,T_e;L^\infty(\Omega))}\leqslant C, \tag{3.18}$$

$$\int_0^t\big\|S_x^\kappa\big\|_{L^\infty(\Omega)}^{\frac83}\,d\tau\leqslant C. \tag{3.19}$$

Proof. For some $2 > p \geqslant 1$ we choose q, q' such that

$$q = \frac{2}{p}, \qquad \frac{1}{q} + \frac{1}{q'} = 1.$$

By Hölder's inequality, we have

$$\begin{aligned}
\int_0^t \int_\Omega \left(\left|S_x^\kappa\right|_\kappa \left|S_{xx}^\kappa\right|\right)^p dx\,d\tau &= \int_0^t \int_\Omega \left(\left|S_x^\kappa\right|_\kappa\right)^{\frac{p}{2}} \left(\left(\left|S_x^\kappa\right|_\kappa\right)^{\frac{p}{2}} \left|S_{xx}^\kappa\right|^p\right) dx\,d\tau \\
&\leqslant \left(\int_0^t \int_\Omega \left(\left|S_x^\kappa\right|_\kappa\right)^{\frac{pq'}{2}} dx\,d\tau\right)^{\frac{1}{q'}} \left(\int_0^t \int_\Omega \left(\left|S_x^\kappa\right|_\kappa\right)^{\frac{pq}{2}} \left|S_{xx}^\kappa\right|^{pq} dx\,d\tau\right)^{\frac{1}{q}} \\
&\leqslant \left(\int_0^t \int_\Omega \left(\left|S_x^\kappa\right|_\kappa\right)^{\frac{p}{2-p}} dx\,d\tau\right)^{\frac{2-p}{2}} \left(\int_0^t \int_\Omega \left|S_x^\kappa\right|_\kappa \left|S_{xx}^\kappa\right|^2 dx\,d\tau\right)^{\frac{p}{2}}.
\end{aligned} \tag{3.20}$$

Estimate (3.5) implies that if p satisfies $\frac{p}{2-p} \leqslant 2$, i.e. $p \leqslant \frac{4}{3}$, then the right-hand side of (3.20) is bounded. This yields estimate (3.14). By definition of $|y|_\kappa$,

$$|y|_\kappa - |y| = \frac{\kappa^2}{|y|_\kappa + |y|}.$$

Since $|y|_\kappa + |y| \geqslant \kappa$, we have

$$\frac{\kappa^2}{|y|_\kappa + |y|} \leqslant \frac{\kappa^2}{\kappa} = \kappa. \tag{3.21}$$

Hence

$$0 \leqslant |y|_\kappa - |y| \leqslant \kappa.$$

Letting $y = S_x^\kappa$ yields

$$\left|S_x^\kappa S_{xx}^\kappa\right| = \left|S_x^\kappa\right|\left|S_{xx}^\kappa\right| \leqslant \left(\left|S_x^\kappa\right|_\kappa - \left|S_x^\kappa\right|\right)\left|S_{xx}^\kappa\right| \leqslant \kappa\left|S_{xx}^\kappa\right|,$$

and (3.15) follows from (3.14) and estimate (3.5).

Next we are going to prove (3.16). Write

$$\left|S_x^\kappa\right|_\kappa S_{xx}^\kappa = \left(\int_0^{S_x^\kappa} |y|_\kappa\,dy\right)_x. \tag{3.22}$$

The primitive of $|y|_\kappa$ is equal to

$$\frac{1}{2}\left(y\sqrt{y^2+\kappa^2} + \kappa^2 \log\left(y + \sqrt{y^2+\kappa^2}\right)\right) + C,$$

thanks to $\log x \leqslant x - 1$ for all $x > 1$ we have

$$\kappa^2 \log\left(y + \sqrt{y^2+\kappa^2}\right) \leqslant C\kappa^2\left(|y|+1\right), \quad \text{if } y \geqslant 1;$$

and noting that $\log x$ is monotone, one has

$$\kappa^2\left|\log\left(y + \sqrt{y^2+\kappa^2}\right)\right| \leqslant C\kappa^2 \max\left\{\log\left(\frac{1}{\kappa}\right),\ \log\left(1+\sqrt{1+\kappa^2}\right)\right\} \leqslant C, \quad \text{for } 0 < y < 1.$$

We then show that

$$\left|\int_\Omega \int_0^{S_x^\kappa} |y|_\kappa\,dy\,dx\right| \leqslant C \int_\Omega \left(\left|S_x^\kappa\right|^2 + 1\right) dx \leqslant C.$$

To apply the Poincaré inequality of the form

$$\|f - \bar{f}\|_{L^p(\Omega)} \leqslant C\|f_x\|_{L^p(\Omega)}$$

where $\bar{f} := \frac{1}{|\Omega|}\int_\Omega f(x)\,dx$, we choose

$$p = \frac{4}{3}, \qquad f = \int_0^{S_x^\kappa} |y|_\kappa\,dy,$$

and obtain

$$\begin{aligned}\int_0^t \left\| \int_0^{S_x^\kappa} |y|_\kappa\,dy \right\|_{L^{\frac{4}{3}}(\Omega)}^{\frac{4}{3}} d\tau &\leqslant C\int_0^t \left\| \left(\int_0^{S_x^\kappa} |y|_\kappa\,dy\right)_x \right\|_{L^{\frac{4}{3}}(\Omega)}^{\frac{4}{3}} d\tau + C\int_0^t \left\| \overline{\int_0^{S_x^\kappa} |y|_\kappa\,dy} \right\|_{L^{\frac{4}{3}}(\Omega)}^{\frac{4}{3}} d\tau \\ &\leqslant C\int_0^t \left\| \left|S_x^\kappa\right|_\kappa S_{xx}^\kappa \right\|_{L^{\frac{4}{3}}(\Omega)}^{\frac{4}{3}} d\tau + C\int_0^t 1\,d\tau, \end{aligned} \tag{3.23}$$

which implies, by (3.14), that

$$\int_0^t \left\| \int_0^{S_x^\kappa} |y|_\kappa\,dy \right\|_{L^{\frac{4}{3}}(\Omega)}^{\frac{4}{3}} d\tau \leqslant C. \tag{3.24}$$

Hence (3.16) follows, and we get $\int_0^{S_x^\kappa} |y|_\kappa\,dy \in L^{\frac{4}{3}}(0, T_e; W^{1,\frac{4}{3}}(\Omega))$. Making use of the Sobolev embedding theorem, we get (3.17).

It remains to prove estimate (3.19), since (3.18) is equivalent to (3.19). We rewrite $\int_0^{S_x^\kappa} |y|_\kappa\,dy$ as

$$\int_0^{S_x^\kappa} |y|_\kappa\,dy = \int_0^{S_x^\kappa} |y|\,dy + \int_0^{S_x^\kappa} \big(|y|_\kappa - |y|\big)\,dy = \frac{1}{2}|y|y\Big|_0^{S_x^\kappa} + \int_0^{S_x^\kappa} \frac{\kappa^2}{|y|_\kappa + |y|}\,dy = \frac{1}{2}\left|S_x^\kappa\right|S_x^\kappa + \int_0^{S_x^\kappa} \frac{\kappa^2}{|y|_\kappa + |y|}\,dy. \tag{3.25}$$

Thus

$$\frac{1}{2}\big(\left|S_x^\kappa\right|S_x^\kappa\big)_x = \left(\int_0^{S_x^\kappa} |y|\,dy\right)_x = \left(\int_0^{S_x^\kappa} |y|_\kappa\,dy\right)_x - \frac{\kappa^2 S_{xx}^\kappa}{|S_x^\kappa|_\kappa + |S_x^\kappa|}. \tag{3.26}$$

By (3.21) and the Young inequality we obtain from (3.5) and the assumption that $k \leqslant 1$ that

$$\left|\frac{\kappa^2 S_{xx}^\kappa}{|S_x^\kappa|_\kappa + |S_x^\kappa|}\right| \leqslant \left|\kappa S_{xx}^\kappa\right|, \quad \text{thus } \left\|\kappa S_{xx}^\kappa\right\|_{L^{\frac{4}{3}}(Q_{T_e})} \leqslant \left(\int_{Q_{T_e}} \left(\kappa^2 + \kappa\left|S_{xx}^\kappa\right|^2\right)dx\,d\tau\right)^{\frac{3}{4}} \leqslant C. \tag{3.27}$$

Combination with (3.16) and (3.26) yields

$$\left\|\big(\left|S_x^\kappa\right|S_x^\kappa\big)_x\right\|_{L^{\frac{4}{3}}(Q_{T_e})} \leqslant C\left\| \left(\int_0^{S_x^\kappa} |y|_\kappa\,dy\right)_x \right\|_{L^{\frac{4}{3}}(Q_{T_e})} + C\left\|\kappa S_{xx}^\kappa\right\|_{L^{\frac{4}{3}}(Q_{T_e})} \leqslant C. \tag{3.28}$$

It is clear that $|\overline{|S_x^\kappa|S_x^\kappa}| \leqslant C\int_\Omega |S_x^\kappa|^2\,dx \leqslant C$. Applying again the Poincaré inequality to the function $f = |S_x^\kappa|S_x^\kappa$, we arrive at

$$\left\|\left|S_x^\kappa\right|S_x^\kappa\right\|_{L^{\frac{4}{3}}(Q_{T_e})} \leqslant C.$$

Hence this, combined with (3.28), implies that

$$\left\|\left|S_x^\kappa\right|S_x^\kappa\right\|_{L^{\frac{4}{3}}(0,T_e;W^{1,\frac{4}{3}}(\Omega))} \leqslant C,$$

one concludes by using the Sobolev embedding theorem that

$$\left\|\left|S_x^\kappa\right|S_x^\kappa\right\|_{L^{\frac{4}{3}}(0,T_e;L^\infty(\Omega))} \leqslant C,$$

which is

$$\left\|S_x^\kappa\right\|_{L^{\frac{8}{3}}(0,T_e;L^\infty(\Omega))} \leqslant C.$$

This completes the proof of this corollary. □

Lemma 3.3. *There hold for any $t \in [0, T_e]$*

$$\left\| S_t^\kappa(t) \right\|^2 + c\nu \int_0^t \int_\Omega \left|S_x^\kappa\right|_\kappa \left|S_{xt}^\kappa\right|^2 dx\, d\tau \leqslant C, \tag{3.29}$$

$$\left\| \left|S_x^\kappa\right|_\kappa S_{xx}^\kappa(t) \right\| \leqslant C. \tag{3.30}$$

Proof. Suppose that estimate (3.29) is true, from Eq. (2.4) and the estimate (3.5) we can easily get (3.30). So it is enough to prove (3.29).

Differentiating (2.4) formally with respect to t yields

$$S_{tt}^\kappa = c\nu\left(\left|S_x^\kappa\right|_\kappa S_{xt}^\kappa\right)_x + c\left(\left(T^\kappa \cdot \bar{\varepsilon} - \hat{\psi}'\left(S^\kappa\right)\right)\left(\left|S_x^\kappa\right|_\kappa - \kappa\right)\right)_t. \tag{3.31}$$

Multiplying (3.31) by S_t^κ and integrating the resulting equation, using integration by parts, we obtain

$$\begin{aligned} 0 &= \frac{1}{2}\frac{d}{dt}\left\|S_t^\kappa\right\|^2 + c\nu \int_\Omega \left|S_x^\kappa\right|_\kappa \left|S_{xt}^\kappa\right|^2 dx + c\int_\Omega \left(\left(T^\kappa \cdot \bar{\varepsilon} - \hat{\psi}'\left(S^\kappa\right)\right)\left(\left|S_x^\kappa\right|_\kappa - \kappa\right)\right)_t S_t^\kappa\, dx \\ &= \frac{1}{2}\frac{d}{dt}\left\|S_t^\kappa\right\|^2 + c\nu \int_\Omega \left|S_x^\kappa\right|_\kappa \left|S_{xt}^\kappa\right|^2 dx + J_1. \end{aligned} \tag{3.32}$$

It is not difficult to carry out a rigorous justification of (3.32) with the help of difference quotient, we omit the details. Computation gives

$$J_1 = c\int_\Omega \left(\left(T^\kappa \cdot \bar{\varepsilon} - \hat{\psi}'\left(S^\kappa\right)\right)_t \left(\left|S_x^\kappa\right|_\kappa - \kappa\right) + \left(T^\kappa \cdot \bar{\varepsilon} - \hat{\psi}'\left(S^\kappa\right)\right)\left(\left|S_x^\kappa\right|_\kappa\right)_t\right) S_t^\kappa\, dx = J_{11} + J_{12}. \tag{3.33}$$

By the formula of T, i.e. (2.10), we have

$$\begin{aligned} |J_{11}| &\leqslant C\int_\Omega \left(\left|S_t^\kappa\right|^2 + \left|S_t^\kappa\right| + 1\right)\left(\left|S_x^\kappa\right|_\kappa + \kappa\right) dx \leqslant C\left(\left\|S_x^\kappa\right\|_{L^\infty(\Omega)} + 1\right)\int_\Omega \left(\left|S_t^\kappa\right|^2 + 1\right) dx \\ &\leqslant C\left(\left\|S_x^\kappa\right\|_{L^\infty(\Omega)} + 1\right)\left(\left\|S_t^\kappa\right\|^2 + 1\right). \end{aligned} \tag{3.34}$$

To handle J_{12}, we make use of estimate (3.6) and $|y| \leqslant |y|_\kappa$.

$$\begin{aligned} |J_{12}| &\leqslant C\int_\Omega \frac{|S_x^\kappa S_{xt}^\kappa S_t^\kappa|}{|S_x^\kappa|_\kappa} dx = C\int_\Omega \frac{|S_x^\kappa S_{xt}^\kappa|}{|S_x^\kappa|_\kappa^{\frac{1}{2}}} \frac{|S_t^\kappa|}{|S_x^\kappa|_\kappa^{\frac{1}{2}}} dx \leqslant C\int_\Omega \frac{|S_x^\kappa|_\kappa |S_{xt}^\kappa|}{|S_x^\kappa|_\kappa^{\frac{1}{2}}} \frac{|S_t^\kappa|}{|S_x^\kappa|_\kappa^{\frac{1}{2}}} dx \\ &\leqslant \frac{c\nu}{2}\int_\Omega \left|S_x^\kappa\right|_\kappa \left|S_{xt}^\kappa\right|^2 dx + C_\nu \int_\Omega \frac{|S_t^\kappa|^2}{|S_x^\kappa|_\kappa} dx. \end{aligned} \tag{3.35}$$

Thus it follows from (3.32)–(3.35) that

$$\frac{1}{2}\frac{d}{dt}\left\|S_t^\kappa\right\|^2 + c\nu\int_\Omega \left|S_x^\kappa\right|_\kappa \left|S_{xt}^\kappa\right|^2 dx \leqslant \frac{c\nu}{2}\int_\Omega \left|S_x^\kappa\right|_\kappa \left|S_{xt}^\kappa\right|^2 dx + C_\nu\int_\Omega \frac{|S_t^\kappa|^2}{|S_x^\kappa|_\kappa} dx + C\left(\left\|S_x^\kappa\right\|_{L^\infty(\Omega)} + 1\right)\left(\left\|S_t^\kappa\right\|^2 + 1\right). \tag{3.36}$$

From Eq. (2.4) and assumption $S_0 \in H^2(\Omega)$ we compute the initial data

$$\begin{aligned} \left\|S_t^\kappa\big|_{t=0}\right\| &\leqslant C\left(\left\||S_{0x}|_\kappa S_{0xx}\right\| + \left\||S_{0x}|_\kappa + \kappa\right\|\right) \leqslant C\left(\left\||S_{0x}|_\kappa\right\|_{L^\infty(\Omega)}\|S_{0xx}\| + \|S_{0x}\| + 1\right) \\ &\leqslant C\left(\left(\|S_{0x}\|_{H^1(\Omega)} + 1\right)\|S_{0xx}\| + \|S_{0x}\| + 1\right) \leqslant C. \end{aligned} \tag{3.37}$$

Thus $S_t^\kappa|_{t=0} \in L^2(\Omega)$. Next we use the Gronwall inequality of the form:

Lemma 3.4. *For measurable functions y, A, B defined on $[0, T_e]$, such that $y \geqslant 0$ and $A, B \in L^1(0, T_e)$, if*

$$y'(t) \leqslant A(t)y(t) + B(t),$$

then

$$y(t) \leqslant y(0)\exp\left(\int_0^t A(\tau)\, d\tau\right) + \int_0^t B(s)\exp\left(\int_s^t A(\tau)\, d\tau\right) ds.$$

Defining

$$y(t)=\left\|S_t^\kappa(t)\right\|^2,\qquad A(t)=C\left(\left\|S_x^\kappa\right\|_{L^\infty(\Omega)}+1\right),\qquad B(t)=C\left(\left\|S_x^\kappa\right\|_{L^\infty(\Omega)}+1\right)+C_\nu\int_\Omega\frac{|S_t^\kappa|^2}{|S_x^\kappa|_\kappa}\,dx,$$

where A, B are integrable over $[0,T_e]$ by Lemma 3.1 and Corollary 3.2, we derive from (3.36) and (3.37) that

$$\left\|S_t^\kappa(t)\right\|^2+c\nu\int_0^t\int_\Omega\left|S_x^\kappa\right|_\kappa\left|S_{xt}^\kappa\right|^2\,dx\,d\tau\leqslant C\left\|S_t^\kappa(0)\right\|^2+C\leqslant C. \tag{3.38}$$

Thus the proof of this lemma is complete. □

Corollary 3.5. *The function $\int_0^{S_x^\kappa}|y|_\kappa\,dy$ belongs to $H^1(Q_{T_e})$, and the estimates hold*

$$\left\|\left(\int_0^{S_x^\kappa}|y|_\kappa^{\frac12}\,dy\right)_t\right\|_{L^2(Q_{T_e})}\leqslant C, \tag{3.39}$$

$$\left\|\int_0^{S_x^\kappa}|y|_\kappa^{\frac12}\,dy\right\|_{L^2(0,T_e;H^1(\Omega))}\leqslant C. \tag{3.40}$$

Proof. Calculating yields

$$\left(\int_0^{S_x^\kappa}|y|_\kappa^{\frac12}\,dy\right)_t=\left|S_x^\kappa\right|_\kappa^{\frac12}S_{xt}^\kappa, \tag{3.41}$$

recalling (3.29), we obtain (3.39). Similarly,

$$\left(\int_0^{S_x^\kappa}|y|_\kappa^{\frac12}\,dy\right)_x=\left|S_x^\kappa\right|_\kappa^{\frac12}S_{xx}^\kappa, \tag{3.42}$$

combining with (3.5) gives

$$\left\|\left(\int_0^{S_x^\kappa}|y|_\kappa^{\frac12}\,dy\right)_x\right\|_{L^2(Q_{T_e})}\leqslant C. \tag{3.43}$$

Finally, noting $|\int_0^{S_x^\kappa}|y|_\kappa^{\frac12}\,dy|\leqslant C\max\{M,|S_x^\kappa|^{\frac32}\}$ for some large constant $M>0$, we have

$$\left\|\int_0^{S_x^\kappa}|y|_\kappa^{\frac12}\,dy\right\|_{L^2(Q_{T_e})}^2\leqslant C+C\int_\Omega\left|S_x^\kappa\right|^3dx\leqslant C+C\left\|S_x^\kappa\right\|_{L^\infty(\Omega)}\int_\Omega\left|S_x^\kappa\right|^2dx. \tag{3.44}$$

Thus by (3.19) in Corollary 3.2 there holds

$$\int_0^t\left\|\int_0^{S_x^\kappa}|y|_\kappa^{\frac12}\,dy\right\|_{L^2(Q_{T_e})}^2d\tau\leqslant C+C\int_0^t\left\|S_x^\kappa\right\|_{L^\infty(\Omega)}d\tau\leqslant C. \tag{3.45}$$

Then (3.40) follows from (3.43) and (3.45). The proof of the corollary is complete. □

4. Existence/regularity of solutions to the phase field model

We shall make use of the a priori estimates established in the previous section to study the convergence of $(u^\kappa,T^\kappa,S^\kappa)$ as $\kappa\to0$. In this section we will show that there is a subsequence, which converges to a weak solution of the initial-boundary value problem (1.7)–(1.12), thus we prove the existence of weak solutions; then we shall investigate the regularity of solutions.

Existence. It follows from Lemmas 3.1 and 3.3 that

$$\left\| S^\kappa \right\|_{H^1(Q_{T_e})} \leqslant C, \tag{4.1}$$

for a constant C independent of κ. Hence, we can select a sequence $\kappa_n \to 0$ and a function $S \in H^1(Q_{T_e})$, such that the sequence S^{κ_n}, which we again denote by S^κ, satisfies

$$\left\| S^\kappa - S \right\|_{L^2(Q_{T_e})} \to 0, \qquad S_x^\kappa \rightharpoonup S_x, \qquad S_t^\kappa \rightharpoonup S_t, \tag{4.2}$$

where the weak convergence is in $L^2(Q_{T_e})$.

Since Eq. (1.9) is nonlinear, the weak convergence of S_x^κ is not enough to prove that the limit function solves this equation. In the following lemma we therefore show that S_x^κ converges pointwise almost everywhere:

Lemma 4.1. *There exists a subsequence of S_x^κ, we still denote it by S_x^κ, such that*

$$\int_0^{S_x^\kappa} |y|_\kappa^{\frac{1}{2}}\,dy \to \int_0^{S_x} |y|^{\frac{1}{2}}\,dy \quad \text{a.e. in } Q_{T_e}, \tag{4.3}$$

$$S_x^\kappa \to S_x, \quad \text{a.e. in } Q_{T_e}, \tag{4.4}$$

$$\left|S_x^\kappa\right|_\kappa \rightharpoonup |S_x|, \quad \text{weakly in } L^2(Q_{T_e}), \tag{4.5}$$

$$\int_0^{S_x^\kappa} |y|^{\frac{1}{2}}\,dy \to \int_0^{S_x} |y|^{\frac{1}{2}}\,dy, \quad \text{strongly in } L^2(Q_{T_e}), \tag{4.6}$$

as $\kappa \to 0$.

The proof is based on the following result:

Lemma 4.2. *Let $(0, T_e) \times \Omega$ be an open set in $\mathbb{R}^+ \times \mathbb{R}^n$. Suppose functions g_n, g are in $L^q((0, T_e) \times \Omega)$ for any given $1 < q < \infty$, which satisfy*

$$\|g_n\|_{L^q((0,T_e)\times\Omega)} \leqslant C, \quad g_n \to g \text{ almost everywhere in } (0, T_e) \times \Omega.$$

Then g_n converges to g weakly in $L^q((0, T_e) \times \Omega)$.

In [2] the existence proof is based on the compactness lemma of Aubin–Lions. Since here we have stronger a priori estimates, we do not need this lemma. Instead we simply rely on the Rellich compactness lemma.

Proof of Lemma 4.1. Since the estimates in Lemma 3.1 and Corollary 3.5 imply that the sequence $\int_0^{S_x^\kappa} |y|_\kappa^{\frac{1}{2}} dy$ is uniformly bounded in $H^1(Q_{T_e})$ for $\kappa \in (0, 1]$. By the Sobolev embedding theorem, we assert that $\int_0^{S_x^\kappa} |y|_\kappa^{\frac{1}{2}}\,dy$ is compact in $L^2(Q_{T_e}) = L^2(0, T_e; L^2(\Omega))$. Thus there is a subsequence, still denoted by $\int_0^{S_x^\kappa} |y|_\kappa^{\frac{1}{2}}\,dy$, which converges strongly in $L^2(Q_{T_e})$ to a limit function $G \in L^2(Q_{T_e})$. Next we prove that the sequence $\int_0^{S_x^\kappa} |y|^{\frac{1}{2}}\,dy$ converges to G in $L^2(Q_{T_e})$. Write

$$\int_0^{S_x^\kappa} |y|^{\frac{1}{2}}\,dy = \int_0^{S_x^\kappa} |y|_\kappa^{\frac{1}{2}}\,dy + \int_0^{S_x^\kappa} \left(|y|^{\frac{1}{2}} - |y|_\kappa^{\frac{1}{2}}\right) dy = I_1 + I_2.$$

It is easy to compute that

$$0 \leqslant |y|_\kappa^{\frac{1}{2}} - |y|^{\frac{1}{2}} = \frac{|y|_\kappa - |y|}{|y|_\kappa^{\frac{1}{2}} + |y|^{\frac{1}{2}}} = \frac{\kappa^2}{(|y|_\kappa^{\frac{1}{2}} + |y|^{\frac{1}{2}})(|y|_\kappa + |y|)} \leqslant \frac{\kappa^2}{\kappa^{\frac{1}{2}+1}} = \kappa^{\frac{1}{2}}. \tag{4.7}$$

Thus I_2 can be estimated as

$$\|I_2\|_{L^2(Q_{T_e})} \leqslant \left\|\kappa^{\frac{1}{2}} S_x^\kappa\right\|_{L^2(Q_{T_e})} \leqslant C\kappa^{\frac{1}{2}} \left\|S_x^\kappa\right\|_{L^\infty(0,T_e;L^2(\Omega))} \leqslant C\kappa^{\frac{1}{2}} \to 0.$$

Therefore, $\int_0^{S_x^\kappa} |y|^{\frac{1}{2}} dy \to \lim_{\kappa\to 0} I_1 = G$ strongly in $L^2(Q_{T_e})$.

Consequently, from this sequence $\int_0^{S_x^\kappa} |y|^{\frac{1}{2}}\,dy$ we can select another subsequence, denoted in the same way, which converges almost everywhere in Q_{T_e}. Using that the mapping $y \mapsto f(y) := \int_0^y |y|^{\frac{1}{2}}\,dy$ has a continuous inverse $f^{-1}: \mathbb{R} \to \mathbb{R}$, we infer that also the sequence $S_x^\kappa = f^{-1}(\int_0^{S_x^\kappa} |y|^{\frac{1}{2}}\,dy)$ converges pointwise almost everywhere to $f^{-1}(G)$ in Q_{T_e}. From the uniqueness of the weak limit we conclude that $f^{-1}(G) = S_x$ almost everywhere in Q_{T_e}. Thus we prove (4.6).

To prove (4.5) we note that the estimate $|S_x^\kappa|_\kappa \leqslant |S_x^\kappa| + \kappa$ and the inequality (4.1) together imply that the sequence $|S_x^\kappa|_\kappa$ is uniformly bounded in $L^2(Q_{T_e})$. Thus, (4.5) is a consequence of (4.4) and Lemma 4.2. $\square$

Proof of Theorem 1.2. Define the functions u, T by inserting S into (2.9) and (2.10), respectively, where S is the limit function of the sequence S^κ. We shall prove that (u, T, S) is a weak solution of problem (1.7)–(1.12).

Recalling (2.8) we have $S \in L^\infty(Q_{T_e})$. From this relation, from the definition of u, T we immediately see that u, T satisfy (1.14) and (1.15), respectively. Observe next that $\|S^\kappa\|_{L^\infty(0,T_e;H_0^1(\Omega))} \leqslant C$, by Lemma 3.1 and Sobolev's embedding theorem. This implies $S \in L^\infty(0, T_e; H_0^1(\Omega))$, since we can select a subsequence of S^κ which converges weakly to S in this space. Thus, S satisfies (1.16).

Noting that from (2.1) and (4.2)

$$\begin{aligned}\|\chi_\kappa * S^\kappa - S\|_{L^2(Q_{T_e})} &\leqslant \|\chi_\kappa * (S^\kappa - S)\|_{L^2(Q_{T_e})} + \|(S - \chi_\kappa * S)\|_{L^2(Q_{T_e})}\\ &\leqslant \|(S - \chi_\kappa * S)\|_{L^2(Q_{T_e})} + \|S^\kappa - S\|_{L^2(Q_{T_e})} \to 0,\end{aligned} \tag{4.8}$$

for $\kappa \to 0$, we conclude easily that the function (u, T) defined in this way satisfy weakly Eqs. (1.7)–(1.8). By definition, if the relation (1.17) holds, then the proof of the existence of weak solutions is complete. To verify (1.17) we use that by construction S^κ solves (2.3). Now we multiply Eq. (2.3) by a test function $\varphi \in C_0^\infty((-\infty, T_e) \times \Omega)$ and integrate the resulting equation over Q_{T_e}, then obtain

$$\begin{aligned}0 &= \left(S_t^\kappa, \varphi\right)_{Q_{T_e}} + \left(-c\nu|S_x^\kappa|_\kappa S_{xx}^\kappa + \mathcal{F}^\kappa\left(|S_x^\kappa|_\kappa - \kappa\right), \varphi\right)_{Q_T}\\ &= -\left(S_0^\kappa, \varphi(0)\right)_\Omega - \left(S^\kappa, \varphi_t\right)_{Q_{T_e}} + \left(c\,\nu \int_0^{S_x^\kappa} |y|_\kappa\,dy, \varphi_x\right)_{Q_{T_e}} + \left(\mathcal{F}^\kappa\left(|S_x^\kappa|_\kappa - \kappa\right), \varphi\right)_{Q_T},\end{aligned}$$

where $\mathcal{F}^\kappa = -c(T^\kappa \cdot \bar{\varepsilon} - \hat{\psi}'(S^\kappa))$. Eq. (1.17) follows from this relation if we show that

$$\left(S_0^\kappa, \varphi(0)\right)_\Omega \to \left(S_0, \varphi(0)\right)_\Omega, \tag{4.9}$$

$$\left(S^\kappa, \varphi_t\right)_{Q_{T_e}} \to (S, \varphi_t)_{Q_{T_e}}, \tag{4.10}$$

$$\left(\int_0^{S_x^\kappa} |y|_\kappa\,dy, \varphi_x\right)_{Q_{T_e}} \to \left(\frac{1}{2}|S_x|S_x, \varphi_x\right)_{Q_{T_e}}, \tag{4.11}$$

$$\left(\mathcal{F}^\kappa|S_x^\kappa|_\kappa, \varphi\right)_{Q_{T_e}} \to \left(\mathcal{F}|S_x|, \varphi\right)_{Q_{T_e}}, \tag{4.12}$$

$$\left(\kappa\mathcal{F}^\kappa, \varphi\right)_{Q_{T_e}} \to 0, \tag{4.13}$$

for $\kappa \to 0$. Now, the relation (4.9) follows from (3.2), and the relation (4.10) is a consequence of (4.2). By (4.4) and (3.30) from which it is easy to get $\|\int_0^{S_x^\kappa} |y|_\kappa\,dy\|_{L^2(Q_{T_e})} \leqslant C$, using again Lemma 4.1, one has (4.11). Convergence (4.13) follows from (3.4) easily.

To verify (4.12) we note that (4.8), (3.19), (3.4), and the definition of $\mathcal{F}^\kappa$ yield

$$\left\|\mathcal{F}^\kappa|S_x^\kappa|_\kappa\right\|_{L^2(Q_{T_e})} \leqslant C, \tag{4.14}$$

$$\mathcal{F}^\kappa|S_x^\kappa|_\kappa \to \mathcal{F}|S_x|, \quad \text{almost everywhere.} \tag{4.15}$$

Then by Lemma 4.1,

$$\mathcal{F}^\kappa|S_x^\kappa|_\kappa \rightharpoonup \mathcal{F}|S_x|,$$

weakly in $L^2(Q_{T_e})$, which implies (4.12). Consequently (1.17) holds.

Regularity. Since $S_0 \in H^2(\Omega)$, we can obtain more regular solutions. By the estimate $\|S_t^\kappa\|_{L^\infty(0,T_e;L^2(\Omega))} \leqslant C$, we see that the relation $S_t \in L^\infty(0, T_e; L^2(\Omega))$ is true. Then by the theory of elliptic systems, we obtain (1.19).

To prove (1.20), we recall the definition of weak solutions. From (1.17) it follows that

$$\begin{aligned} \left|\big(|S_x|S_x,\varphi_x\big)_{Q_{T_e}}\right| &\leqslant C\left|\big(\big(T\cdot\bar{\varepsilon}-\hat{\psi}'(S)\big)|S_x|,\varphi\big)_{Q_{T_e}}\right| + \left|(S,\varphi_t)_{Q_{T_e}} + \big(S_0,\varphi(0)\big)_{\Omega}\right| \\ &\leqslant C\|S_x\|_{L^\infty(0,T_e;L^2(\Omega))}\|\varphi\|_{L^1(0,T_e;L^2(\Omega))} + \left|(S_t,\varphi)_{Q_{T_e}}\right| \\ &\leqslant C\|\varphi\|_{L^1(0,T_e;L^2(\Omega))} + \|S_t\|_{L^\infty(0,T_e;L^2(\Omega))}\|\varphi\|_{L^1(0,T_e;L^2(\Omega))} \\ &\leqslant C\|\varphi\|_{L^1(0,T_e;L^2(\Omega))}, \end{aligned} \tag{4.16}$$

here, we used the estimates $\|S_x\|_{L^\infty(0,T_e;L^2(\Omega))} \leqslant C$ and $\|S_t\|_{L^\infty(0,T_e;L^2(\Omega))} \leqslant C$. The right-hand side of (4.16) holds for all $\varphi \in L^1(0,T_e;L^2(\Omega))$, whence

$$\sup_{0\leqslant t\leqslant T_e}\left\|\big(|S_x|S_x\big)_x(t)\right\| = \sup_{\|\varphi\|_{L^1(0,T_e;L^2(\Omega))}\leqslant 1}\left|\big(\big(|S_x|S_x\big)_x,\varphi\big)_{Q_{T_e}}\right| = \sup_{\|\varphi\|_{L^1(0,T_e;L^2(\Omega))}\leqslant 1}\left|\big\langle|S_x|S_x,\varphi_x\big\rangle_{Q_{T_e}}\right| \leqslant C. \tag{4.17}$$

Thus, $(|S_x|S_x)_x \in L^\infty(0,T_e;L^2(\Omega))$.

Furthermore, from the Poincaré inequality and the estimate $\|S_x\|_{L^\infty(0,T_e;L^2(\Omega))} \leqslant C$ we obtain $|S_x|S_x \in L^\infty(0,T_e;H^1(\Omega))$, from which one asserts by the Sobolev embedding theorem that $|S_x|S_x \in L^\infty(Q_{T_e})$, hence $S_x \in L^\infty(Q_{T_e})$. And the proof of Theorem 1.2 is complete. □

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